Critical onset in coherent oscillations between two weakly coupled Bose-Einstein condensates

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The Josephson effects in two weakly linked Bose-Einstein condensates have been studied recently. In this letter, we study the equations derived by Giovanazzi et. al. [Phys. Rev. Lett. **84**, 4521 (2000)] focusing on the effects of the initial acceleration and the velocity of the barrier on the "dc" current. We find that the dc current has lifetime which critically depends on the moving velocity of the barrier. Moreover, the influence of the initial acceleration is also investigated and found to be crucial for the experimental realization of the effects.

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The Josephson effects (JE's) as a paradigm of the phase coherence manifestation in a macroscopic quantum system, have been observed in superconductors [1], and demonstrated in two weakly linked superfluid ^{3}He-B reservoirs [2]. Since magnetic and optical traps can be tailored and biased with high accuracy [3–5], the weakly interacting Bose-Einstein condensate (BEC) can provide a further context for JE's and reveal novel properties that might not be accessible with other systems. Recently the dc and ac Josephson effects in two weakly linked BECs have been extensively studied [6,8]. These authors suggested that as the barrier between the two trapped BECs moves adiabatically across the trapping potential, a dc current of atoms between two condensates can be found. In analog of the voltage-current characteristic in superconducting Josephson junction (SJJ), there exists a critical velocity of the barrier, at which an abrupt transition from the dc to ac current occurs. In this letter, we study the model introduced in Ref. [6] focusing on the effects of the initial acceleration and the velocity of the barrier on the "dc" current. We find that the dc current has lifetime which depends on the velocity of the barrier and is sensitive on the choice of the initial conditions. To consider the experimental observability of this phenomenon, we investigate the influence of the initial acceleration and find it plays a crucial role.

The interacting BECs in a trap at zero temperature can be described by a macroscopic wave function $\Psi(\mathbf{r},t)$, having the meaning of an order parameter and satisfying the Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = [H_0 + g|\Psi(\mathbf{r},t)|^2]\Psi(\mathbf{r},t), \qquad (1)$$

where $H_0 = -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t)$ and $g = 4\pi\hbar^2 a/m$ with m the atomic mass, and a the s-wave scattering length of the atoms. Considering the system proposed in Ref. [6], a double-well trap produced by a far off-resonance laser barrier, $V_{laser}(z) = V_0 \exp[-(z-l)^2/\lambda^2]$, which cuts a single trapped condensate into two parts [7]. So, the external potential is given by the magnetic trap and the laser barrier $V_{ext}(\mathbf{r},t) = V_{trap}(\mathbf{r}) + V_{laser}(z,t)$.

By solving variationally the GPE using the ansatz: $\Psi(\mathbf{r},t) = \psi_1(t)\phi_1(\mathbf{r}) + \psi_2(t)\phi_2(\mathbf{r})$, where $\psi_{1,2} = \sqrt{N_{1,2}(t)}e^{i\theta_{1,2}(t)}$ are complex time-dependent amplitudes, $N_{1,2}(t)$ and $\theta_{1,2}$ are the number of atoms and the phase of the two condensates respectively. The trial wave functions $\phi_{1,2}(\mathbf{r})$ are orthonormal and can be interpreted as approximate ground state solutions of the GPE of two well respectively. Then the equations of the motion for the relative population $p(t) = (N_1(t) - N_2(t))/N$ and phase $\theta = \theta_2 - \theta_1$ between the two condensates should be [6,8]

$$\dot{\theta} = -Fl + \frac{Kp}{\sqrt{1-p^2}}\cos\theta + Cp, \qquad (2)$$

$$\dot{p} = -K\sqrt{1-p^2}\sin\theta, \qquad (3)$$

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where $K=-\frac{2}{\hbar}\int d\mathbf{r}\phi_1[H_0+gN\phi_1^2]\phi_2$, $C=\frac{gN}{\hbar}\int|\phi_1|^4d\mathbf{r}$ and $F=\frac{1}{\hbar}\int d\mathbf{r}[\phi_1^2-\phi_2^2]\frac{\partial}{\partial l}V_{laser}$. These equations describe the dynamics of the Bose Josephson junction (BJJ). For the convenience we set the parameters as the same as in Ref. [6], $K=4.82\times10^{-4}\mathrm{ms}^{-1}$, $C=1.23\ \mathrm{ms}^{-1}$ and $F=1.06\ \mathrm{ms}^{-1}\mu\mathrm{m}^{-1}$.

The Josephson current is defined as the current of atoms across the barrier, which is $J=\dot{N}_1=-\dot{N}_2$. Considering the total number of atoms is constant , we introduce the normalized current $j=J/N$. From the Eq. (3), we have

$$j=\frac{\dot{p}}{2}=-\frac{K}{2}\sqrt{1-p^2}\sin\theta. \tag{4}$$

Firstly, let us review the properties of this system for l is fixed [8]. For this case, p and θ are canonically conjugate variable of a classical Hamiltonian $H=\frac{C}{2}p^2-K\sqrt{1-p^2}\cos\theta-Flp$, with $\dot{p}=-\frac{\partial H}{\partial\theta}$, $\dot{\theta}=\frac{\partial H}{\partial p}$. This system exhibits two qualitatively different orbits: rotation and libration . In the rotation regime, θ increases (or decreases) monotonically and p oscillates with small amplitude. In the libration regime, θ and p oscillates around the equilibrium point $P_e(p_e,\theta_e)$. The separatrix between the two regimes determined by the saddle point $P_s(p_s,\theta_s)$ with $H_s=H(p_s,\theta_s)$. For large energy ($H>H_s$) the orbit is rotation, whereas for small energy ($H<H_s$) the orbit is libration. The equilibrium point and the saddle point are obtained by equating the right hand sides of Eqs. (2) and (3) to zero, yielding $p_e\approx\frac{Fl}{C},\theta_e=0$ and $p_s\approx\frac{Fl}{C},\theta_s=\pi$.

In the rotation regime, since θ is increasing (or decreasing) monotonically with the angular frequency proportion to $\sqrt{KC}$, from equation (4), we find that j is a fast oscillation current with frequency proportion to $\sqrt{KC}$ and the upper bound of the current, i.e., the so-called critical current $j_c\approx\frac{K}{2}\sqrt{1-p_a^2}$ where p_a is the average of p.

In the liberation regime, let $p=p_e+\delta p$, considering $C\gg K$, from Eqs. (2) and (3) we can obtain a pendulum equation: $\ddot{\theta}+KC\sqrt{1-p_e^2}\sin(\theta)=0$. Then, we obtain $\theta=\theta_p\sin(\omega t)$, $\delta p=\frac{\omega\theta_p}{C}\cos(\omega t)$ in which $\omega=\sqrt{KC(1-p_e^2)}$ and θ_p is determined by the energy of the pendulum. So, one gets $\delta p=\theta_p\sqrt{\frac{K}{C}(1-p_e^2)}\sim\sqrt{\frac{K}{C}}\approx10^{-2}$. For this case, the Josephson current is $j=-\frac{K}{2}\sqrt{1-p_e^2}\sin[\theta_p\sin(\omega t)]$. The critical current is $j_c=\frac{K}{2}\sqrt{1-p_e^2}\sin(|\theta_p|)$ or $j_c=\frac{K}{2}\sqrt{1-p_e^2}$ when $|\theta_p|\geq\pi/2$.

From the above discussion, for a fixed l one can only find the ac current in BJJ. These properties exhibit the analog of the dc voltage case in SJJ.

As it has been suggested in Ref. [6], a dc current can be induced by moving the laser barrier across the trap with a constant velocity $V=\frac{dl}{dt}$, and exhibit the analog of the critical behavior in SJJ's. Now the question arises: can this system exhibit some new properties which can not find in SJJ? To answer this question, let us review the case for SJJ. In SJJ, we know that the densities of cooper pair in two side of the junctions are equal to each other (if the materials are the same) and should hardly change when the dc current exists. This feature interpreting to the BJJ's is that the change in the relative population is very small. To see this feature clearly, let us assume the initial relative population $p(0)=0$, since the change in the relative population is very small, the Eqs. (2) and (3) can be approximated to

$$\dot{p}=-K\sin(\theta) \tag{5}$$

$$\dot{\theta}=-FVt+Cp. \tag{6}$$

Differentiating the second of these equations and replacing the first, we have $\ddot{\theta}+KC\sin\theta=-FV$, which is a driven pendulum equation. The first integral of the equation gives the energy of this pendulum $E=\frac{1}{2}\dot{\theta}^2-KC\cos\theta+FV\theta$, and the constant $E=\frac{1}{2}\dot{\theta}(0)^2-KC\cos\theta[(0)]+FV\theta(0)$. We define the potential

$$U(\theta)=-KC\cos\theta+FV\theta. \tag{7}$$

When $V<0.559\mu m/s$, i.e., $FV/(KC)<1$, it is a washboard potential, its local maximum appears at $\theta=(2m-1)\pi+\arcsin k$, and the local minimum appears at $\theta=2m\pi-\arcsin k$, where $k=\frac{FV}{KC}$ and m is an integer. We restrict the initial θ in interval $[-\pi,\pi]$, then the motion of θ is characterized by the local maximum $U_c=KC\sqrt{1-k^2}+FV[\arcsin k-\pi]$. If $E<U_c$, the motion of θ is an oscillation in the interval $[\theta_{\max},\theta_{\min}]$ with the frequency $\omega\approx\sqrt{KC}$, where $\theta_{\max}$ and $\theta_{\min}$ is the solution of $E=U(\theta)$. If $E>U_c$, the motion of θ is a rotation, i,e., θ decreases monotonically, and $\dot{\theta}=\sqrt{2(E+KC\cos\theta-FV\theta)}$.

When $V > 0.559\mu m/s$, i.e., $FV/(KC) > 1$, the potential is a titled-step potential, θ is also a rotation with $\dot{\theta} \sim -FVt$. From (4), we know it is an ac current when θ is a rotation.

To investigate the case when θ is in oscillation regime, let $p = p_e(t) + \delta p$ where $p_e(t) = \frac{FV}{C}t$, from (6), we obtain $\delta p \approx \frac{\dot{\theta}}{C}$, so, we know that δp is a small term with $\delta p \sim \sqrt{\frac{K}{C}} \approx 0.02$. Then from (4) we can obtain a dc current (to zero order of δp)

$$j = \frac{FV}{2C}. \tag{8}$$

From the above discussion, when θ is in oscillation regime, the current is a dc current with the relative population increasing ($p \approx p_e(t)$); when θ is in rotation regime, the current is an ac one. The motion of θ is determined by the initial conditions and the driven force $\frac{FV}{C}$. If sets the initial conditions: $\theta(0) = \dot{\theta}(0) = 0$, one can obtain a critical velocity $V_c = 0.406\mu m/s$. For $V < V_c$, then $E < U_c$, it is a dc current, but for $V > V_c$,then $E > U_c$, the current is in ac regime. This is a close analog of the critical behavior in SJJ.

However, for the BJJ's, the dc current must lead to the change of the relative population. This feature will give rise to new properties. Taking the change of the relative population into account, we obtain a nonrigid driven pendulum equation

$$\ddot{\theta} + KC\sqrt{1-p^2}\sin\theta = -FV. \tag{9}$$

In analog to the above discussion, we get the potential

$$U(p,\theta) = -KC\sqrt{1-p^2}\cos\theta + FV\theta. \tag{10}$$

This is also a tilted washboard potential for $V < 0.559\mu m/s$. The new feature is the local maximum $U_c(p)$ will decrease with $|p|$ increasing. If the motion of θ is an oscillation at the beginning ($U_c(p) > E$), we will firstly find a dc current with $U_c(p)$ decreasing. The oscillation of θ will not keep when $U_c(p) \leq E$, at this moment θ will start rotating, and then the current will transition to an ac one. This means the dc current does not keep for all the time, i.e., the dc current has a lifetime. So one can have a straightforward definition of lifetime of a dc current as: the lifetime of the dc current τ_c is the time at which $E \geq U_c(p(\tau_c))$. This definition is consistent with the definition of the critical velocity in SJJ's.

The voltage-current characteristic is the most important property in SJJ's. But in BJJ, the important physical quantity should be the relative population p which can be directly detected. So, the critical behavior should be characterized by the change of p in BJJ. When θ is in oscillation regime, from the above discussion the relative population is increasing. When θ is rotating, if $\theta < (\arcsin k - \pi)$, $\dot{\theta} \approx \sqrt{2(E + KC\cos\theta - FV\theta)}$, δp is still a small term with $\delta p_{\max} \sim \sqrt{\frac{K}{C}} \approx 0.02$, hence the relative population is still increasing and $p \approx p_e(t)$ (in the zero order of δp). But if $\theta > (\arcsin k - \pi)$, $\dot{\theta}$ will decrease monotonically, $\dot{\theta} \approx -FVt$, then $\theta \approx -\frac{1}{2}FVt^2$, the integral of the current: $\int_{t_0}^{\infty} K\sin(\frac{1}{2}FVt^2)dt \approx 0.0175$ where t_0 is a finite time, so p should hardly increase. This means that when the motion of θ changes from oscillation to rotation, the relative population will still increase until $\theta \geq (\arcsin k - \pi)$. So, the more accurate definition of the lifetime of the dc current in BJJ should be: the lifetime τ_p is the time when $\theta(\tau_p) \geq (\arcsin k - \pi)$, within this time the relative population is increasing, but after this time the relative population will hardly increase and keep on average fixed.

In Fig. 1, we show the phase diagram relating to the critical onset in the parameter space of t and the laser velocity V for the initial conditions: $l(0) = 0$, $p(0) = 0$ and $\theta(0) = 0$. The solid line is the lifetime $\tau_p(V)$, the dashed line is the lifetime $\tau_c(V)$. The step structure implies that the transitions occur in different cycle of the oscillation. The abruptly increase is due to that the transition occurs near the peak of the potential where $\dot{\theta} \approx 0$. Fig. 2. plots the relative population after the dc current is destroyed, the crosses are the relative population p at $t = 2s$ which are obtained by integrating the Eqs (2) and (3), the solid line is the theoretical result given by $p = p_e(\tau_p)$, which shows the theoretical estimation is consistent with the numerical simulation.

From the above discussion, we know that the lifetime of a dc current is also determined by the initial conditions. One knows that the initial relative population $p(0)$ must be very close to the equilibrium p_e to obtain a dc current, but the initial relative phase can be various. In

Fig. 3, we plot the lifetime of the dc current for different initial relative phase $\theta(0)$, where we let the initial population $p(0) = 0$ and $l(0) = 0$.

Concerning a possible realization of the phenomenon described in this work, one should consider the influence on the initial accelerations. We choose the initial conditions as: $l(0) = 0$, $p(0) = 0, \theta(0) = 0$ and $V(0) = 0$. Let the barrier starts moving with a constant initial acceleration. When its velocity reaches the value V_0, we stop accelerating and then keep the velocity. What we concern is the change on the lifetime of dc current caused by the different initial acceleration. Fig. 4(a) shows the lifetime for different initial acceleration, where the lifetime is the time duration of a dc current after the acceleration. The solid line is for $V_0 = 0.5$ $\mu m/s$ and the dashed line is for $V_0 = 0.4$ $\mu m/s$. We find that for a sudden change of the initial barrier, the influence of the initial accelerations can be negligible, whereas for slow acceleration process the lifetime changes abruptly. The reason can be given by the following analysis: For a sudden acceleration, the time for accelerating is very short. In this time period , the change of θ is small, so does the change of lifetime. On the contrary, in a slowly acceleration process, the θ changes greatly (see Fig. 4(b)). Because the lifetime is very sensitive to the initial conditions, so the lifetime will change dramatically.

We note that the conserved conditions implies that l must be less than $\frac{C}{F} \approx 1.16\mu m$ to ensure $p_e < 1$. In another aspect, to ensure the number of the condensed atoms in one well beyond the minimum threshold, l has to be less than this value. On the other hand, although the Eqs. (2) and (3) is obtained by solving variationally the GPE (1) under the approximation $p << 1$, it still remains a good approximation even for $p \approx 0.4$ [6]. So, the discussion in this paper is accurate at least under these constraints.

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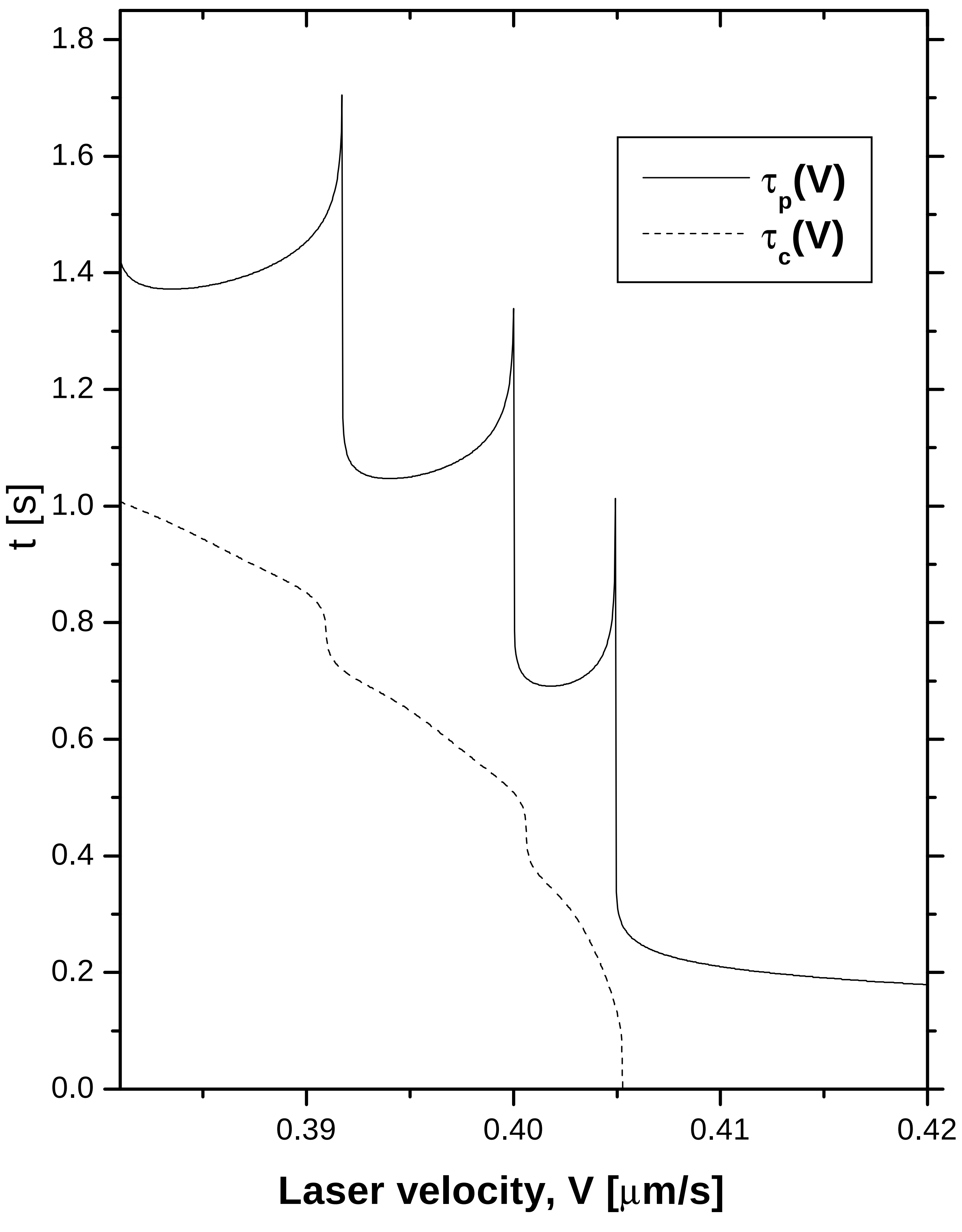
FIGURES CAPTION

Fig. 1. The phase diagram relating to the critical onset in the parameter space of t and the laser velocity V for the initial conditions: $l(0) = 0$, $p(0) = 0$ and $\theta(0) = 0$. The solid line is the lifetime $\tau_p(V)$, the dashed line is $\tau_c(V)$.

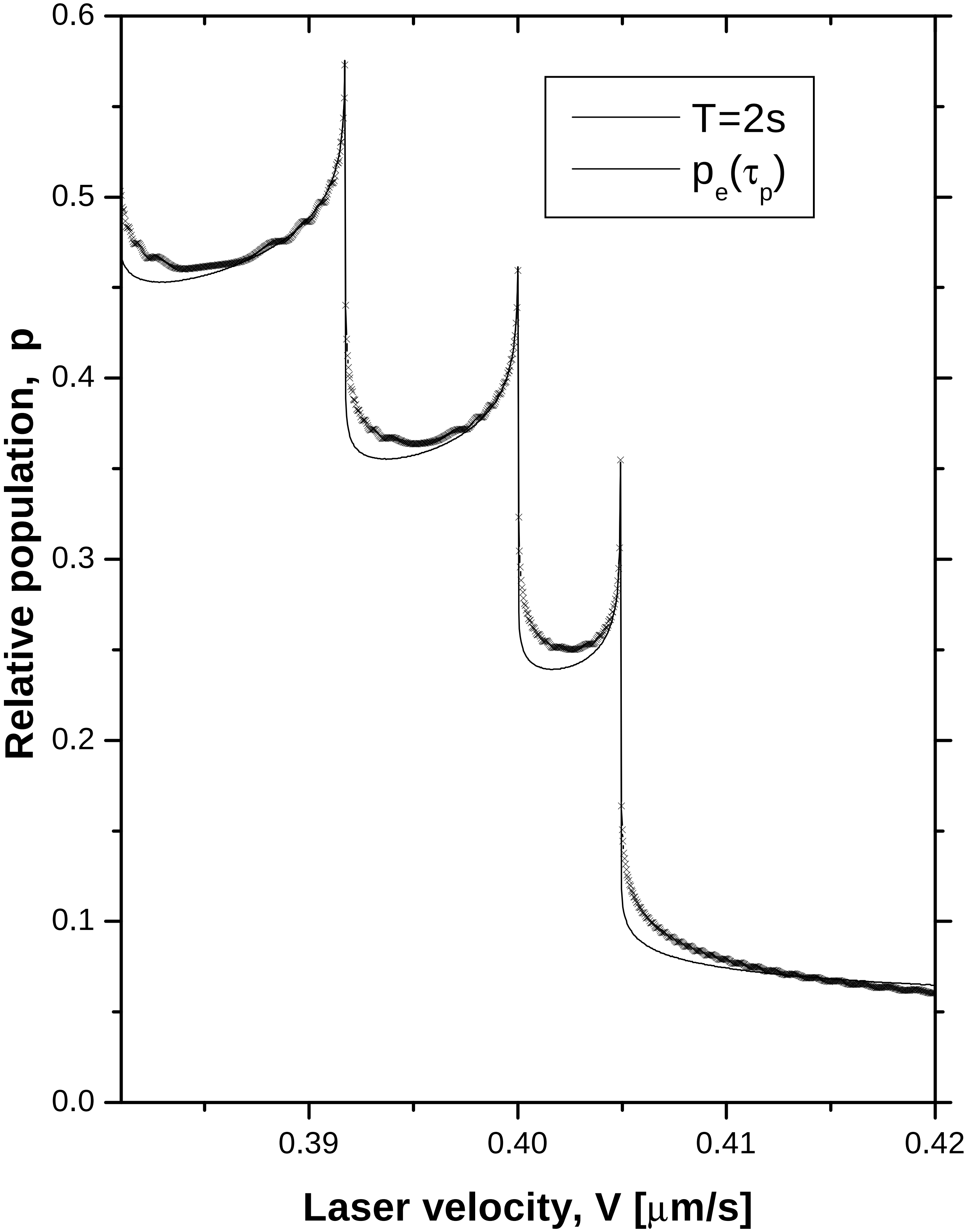
Fig. 2. The relative population after the dc current is destroyed.

Fig. 3. The lifetime of the dc current for different initial θ. The initial relative population is $p(0) = 0$ and $l(0) = 0$.

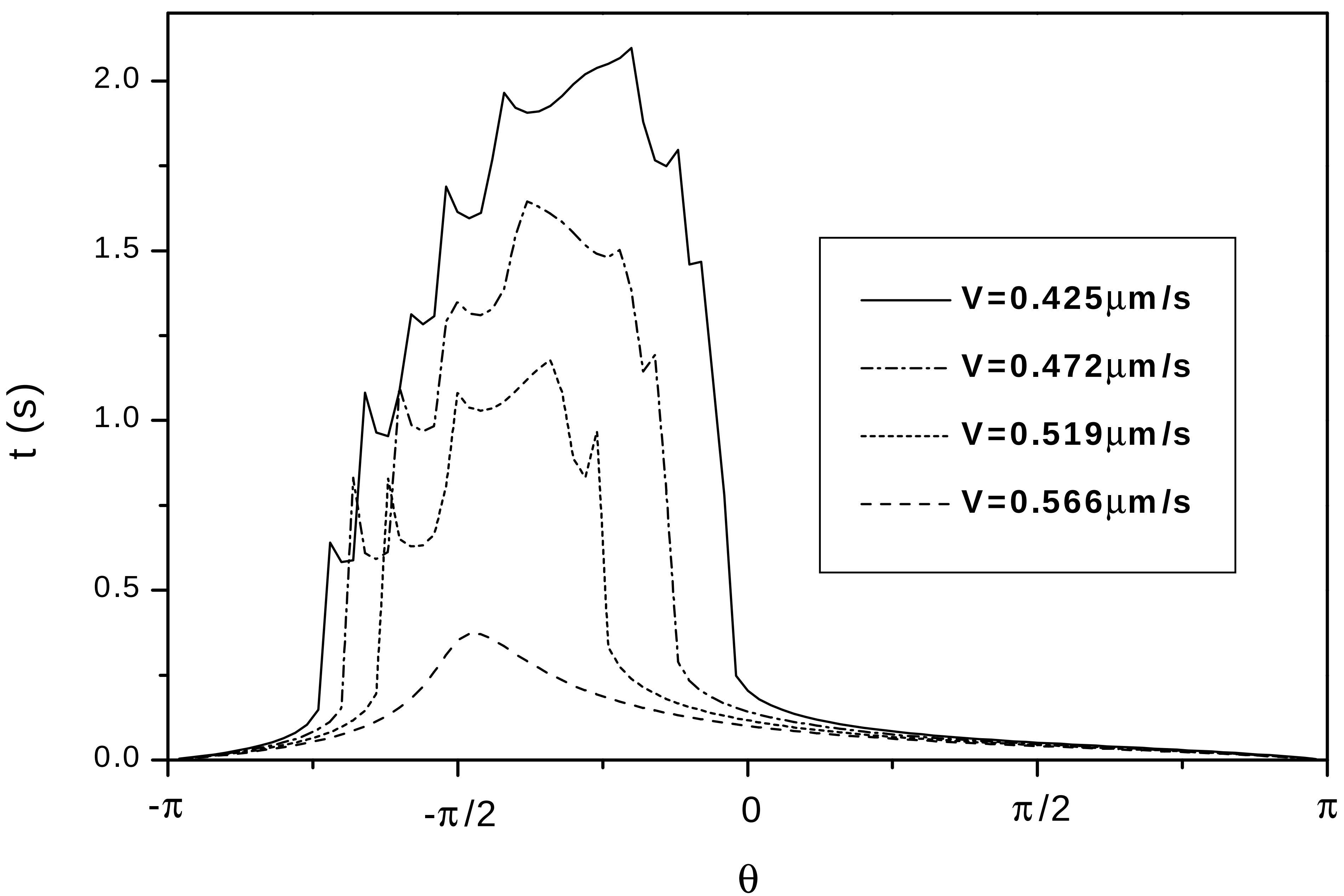
Fig. 4(a). The lifetime for different initial accelerations. The solid line is for $V_0 = 0.5$ $\mu m/s$ and the dashed line is for $V_0 = 0.4$ $\mu m/s$. (b). The relative phase θ when the velocity reaches V_0.



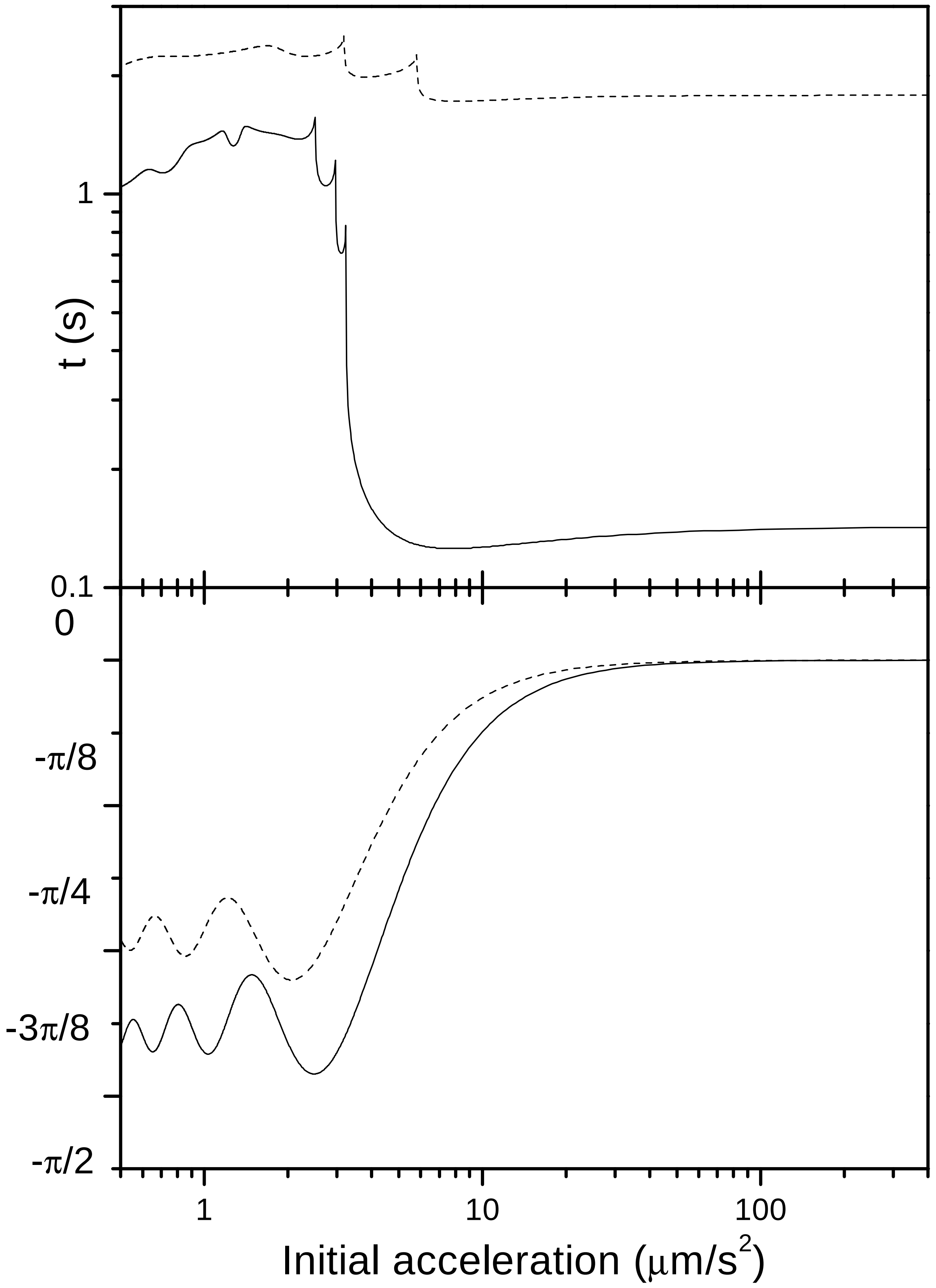
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
t [s]
$\tau_p(V)$
$\tau_c(V)$
0.39
0.40
0.41
0.42
Laser velocity, V [μm/s]



0.6
0.5
0.4
0.3
0.2
0.1
0.0
Relative population, p
0.39
0.40
0.41
0.42
Laser velocity, V [μm/s]
T=2s
$p_e(\tau_p)$



t (s)
2.0
1.5
1.0
0.5
0.0
-π
-π/2
0
π/2
π
θ
V=0.425μm/s
V=0.472μm/s
V=0.519μm/s
V=0.566μm/s


t (s)
1
0.1
0
-π/8
-π/4
-3π/8
-π/2
1
10
100
Initial acceleration (μm/s²)